TWO THEOREMS ON THE VANISHING OF EXT

ABSTRACT. We prove two theorems on the vanishing of Ext over commutative Noetherian local rings. Our first theorem shows that there are no Burch ideals which are rigid over non-regular local domains. Our second theorem reformulates a conjecture of Huneke-Wiegand in terms of the vanishing of Ext, and highlights its relation with the celebrated Auslander-Reiten conjecture. We also discuss several consequences of our results, for example, about the rigidity of the Frobenius endomorphism in prime characteristic p and a generalization of a result of Araya.

1. INTRODUCTION

Throughout, unless otherwise stated, R denotes a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k, and modules over R are assumed to be finitely generated.

An R-module M is called *rigid* if $\mathrm{Ext}^1_R(M,M)=0$. We say M has *rank* provided that there is an integer $r \geq 0$ such that $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus r}$ for all $\mathfrak{p} \in \mathrm{Ass}(R)$, that is, for all associated primes $\mathfrak{p}$ of R. The *torsion submodule* of M is the kernel of the natural homomorphism $M \to \mathrm{Q}(R) \otimes_R M$, where $\mathrm{Q}(R)$ is the total quotient ring of R. The module M is called *torsion-free* if its torsion submodule is zero, and *torsion* if its torsion submodule equals itself. It follows that M is torsion-free if and only if every non zero-divisor on R is a non-zerodivisor on M, and M is torsion if and only if $M_{\mathfrak{p}}=0$ for all $\mathfrak{p} \in \mathrm{Ass}(R)$; see [19, 3.8].

In this paper we are concerned with the Gorenstein case of a long-standing conjecture of Huneke and Wiegand; see [31, page 473] and also [12, 8.6].

Conjecture 1.1. (Huneke-Wiegand) Let R be a one-dimensional Gorenstein ring and let M be a torsion-free R-module that has rank. If M is rigid, then M is free.

The rank and one-dimensional hypotheses are necessary for Conjecture 1.1 which is wide open in general, even for complete intersection rings of codimension two; see, for example, [12]. On the other hand there are affirmative answers over several classes of rings and for quite a few examples of modules. For example, a result of Huneke-Wiegand [31, 3.1], in view of [10, 2.13], implies that Conjecture 1.1 is true over hypersurface rings. We refer the reader to [9, 10, 12, 17, 21, 26, 29, 37] for details and further examples concerning the conjecture.

The aim of this paper is twofold: we prove two theorems, namely Theorem 1.2 and Theorem 1.5, and make progress on Conjecture 1.1 from two different perspectives. As a byproduct, we also generalize a theorem of Araya which considers maximal Cohen-Macaulay modules over Gorenstein rings; see A.1.

Dey and Kobayashi [17, 3.1] extended the definition of a Burch ideal [15] to a Burch module: an R-module M is said to be *Burch* if it is an R-submodule of an R-module T such that $\mathfrak{m}(M :_T \mathfrak{m}) \nsubseteq \mathfrak{m}M$. As each Burch ideal is a Burch module, examples of Burch modules are abundant in the literature; see 2.4. Burch ideals, Burch modules, and Burch rings, as well as related topics, have recently garnered significant attention; see [13, 15, 16, 22]

Our first theorem establishes Conjecture 1.1 for Burch modules and show that, over one-dimensional non-regular local domains, there are no ideals that are both Burch and rigid. In fact our result is more general and shows the following; see Theorem 2.14.

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Theorem 1.2. *Let M be a torsion-free R-module which has rank. If M is Burch and rigid, then M is free, and R is a field or a discrete valuation ring.*

In view of Theorem 1.2, it is worth noting the following: if M is a Burch module over a ring R, then the vanishing of $\operatorname{Ext}_R^{i+1}(M,M)$ and $\operatorname{Ext}_R^{i+2}(M,M)$ for some $i \geq 0$ implies that $\operatorname{pd}_R(M) < \infty$; see [17, 3.16(2)]. However, if M is a Burch module over R, even if M is an ideal and R is a one-dimensional domain, it is not known, in general, whether the vanishing of $\operatorname{Ext}_R^j(M,M)$ for some $j \geq 2$ forces M to have finite projective dimension.

If R and M are as in Conjecture 1.1, it follows that M is rigid if and only if $M \otimes_R M^*$ is torsion-free, where $M^* = \operatorname{Hom}_R(M,R)$; see 3.5(i). Hence, as a consequence of Theorem 1.2, we obtain:

Corollary 1.3. *Let R be a one-dimensional Gorenstein ring and let M be an R-module which has rank. Assume M and $M \otimes_R M^*$ are both torsion-free. If M is Burch, then R is regular and M is free.*

If R is a one-dimensional F-finite Cohen-Macaulay local ring of prime characteristic p, and if ${}^eR$ is rigid for some $e \gg 0$, then it follows from [45, 6.10] that R is a discrete valuation ring. As a consequence of Theorem 1.2, we are able to extend this result and prove the following corollary in section 2; see also 2.4(vi) and Remark 2.16.

Corollary 1.4. *Let R be an equi-dimensional reduced ring of prime characteristic p with* $\operatorname{depth}(R) = 1$. *Assume R is F-finite, for example, R is excellent and k is perfect. If ${}^eR$ is rigid for some $e \gg 0$, then R is a discrete valuation ring.*

The commutative version of the celebrated Auslander-Reiten conjecture [6] claims that each R-module M must be free if $\operatorname{Ext}_R^i(M,M) = \operatorname{Ext}_R^i(M,R) = 0$ for all $i \geq 1$. It is known that, if Conjecture 1.1 holds, then the Auslander-Reiten conjecture holds over each Gorenstein local domain (of arbitrary dimension); see [12, 8.6] for the details. The second theorem we prove generalizes this fact and shows that Conjecture 1.1 implies a stronger conclusion over Gorenstein local domains:

Theorem 1.5. *The following conditions are equivalent:*

(a) *Conjecture 1.1 holds over each (one-dimensional Gorenstein local) domain.*
(b) *Whenever R is a d-dimensional Gorenstein domain and M is a torsion-free R-module such that* $\operatorname{Ext}_R^i(M,M) = 0$ *for all $i = 1, \dots, d$, it follows that M is free.*

In view of the foregoing discussion, Theorem 1.5 yields the following observation:

Corollary 1.6. *The Auslander-Reiten conjecture holds over each Gorenstein local domain (of arbitrary dimension) if the following condition holds: Whenever R is a d-dimensional Gorenstein domain and M is a torsion-free R-module such that* $\operatorname{Ext}_R^i(M,M) = 0$ *for all $i = 1, \dots, d$, it follows that M is free.*

Let us note the following result concerning Theorem 1.5: if R is a d-dimensional Gorenstein local ring with $d \geq 1$, M is a torsion-free R-module such that $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of height at most one, and $\operatorname{Ext}_R^i(M,M) = 0$ for $i = 1, \dots, d-1$, then M is free; see Corollary A.4.

The main ingredient of the proof of Theorem 1.5 is Proposition 3.2. A byproduct of that proposition, along with a result of Kimura [32], allows us to prove the following in the appendix.

Proposition 1.7. *Let R be a ring such that $d =$* $\operatorname{depth}(R) \geq 1$ *and let M be an R-module. Assume the following hold:*

(i) $\operatorname{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$.
(ii) *M is torsion-free.*
(iii) $\operatorname{G-dim}_R(M) < \infty$

If $\operatorname{Ext}_R^{d-1}(M,M) = 0$, *then* $\operatorname{pd}_R(M) \leq d-2$.

Proposition 1.7 was initially proved by Araya [1] for the case where R is Gorenstein and M is maximal Cohen-Macaulay; see 3.1 and A.1. We also prove a variation of Proposition 1.7 by replacing the finite Gorenstein dimension hypothesis on M with the assumption of the vanishing of $\operatorname{Ext}^i_R(M,R)$ for all $i = d, \dots, 2d+1$ under mild additional conditions; see Proposition A.6 and Corollary A.7.

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. First we recall some examples of Burch ideals and modules, and prepare auxiliary results which play an important role in the proof of the theorem.

2.1. Let M be an R-module. Then M is said to be a *Burch* R-submodule of an R-module T provided that $\mathfrak{m}(M :_T \mathfrak{m}) \nsubseteq \mathfrak{m}M$; if such a module T exists, we simply call M a Burch R-module; see [17, 3.1]. Note that, by definition, a Burch module is nonzero.

We need the following properties in the sequel:

2.2. Let M be an R-module.

(i) If $\operatorname{depth}_R(M) \geq 1$, then M is Burch if and only if there is an $x \in \mathfrak{m}$ such that x is a non zero-divisor on M and k is a direct summand of M/xM; see [17, 3.6 and 3.9].
(ii) If M is Burch and $\operatorname{pd}_R(M) < \infty$ (or $\operatorname{id}_R(M) < \infty$), then R is regular; see [17, 3.19].

There are many examples of Burch ideals and modules. Here we record a few of them:

Example 2.3. If $R = k[[t^4,t^5,t^6]]$ and $I = (t^{17},t^{19},t^{20})$, or $R = k[[x,y]]$ and $I = (x^5,x^3y,xy^3,y^5)$, then I is Burch; see [11, 4.4 and 4.5] and also cf. Example 2.18.

2.4. Let I be an ideal of R and let M be an R-module.

(i) If $\operatorname{depth}(R) \geq 1$ and I is integrally closed, then I is Burch; see [11, 2.2(4)].
(ii) If R is not a field, $|k| = \infty$, $\operatorname{depth}(R/I) = 0$, and I is integrally closed, then I is Burch; see [22, 4.6].
(iii) If M is an R-submodule of an R-module T such that $(\mathfrak{m}M :_T \mathfrak{m}) \subseteq M$ and $\operatorname{depth}_R(T/M) = 0$, then M is a Burch submodule of T; see [17, 4.3].
(iv) If $\mathfrak{m}M \neq 0$, for example, if $\operatorname{depth}_R(M) \geq 1$, then $\mathfrak{m}M$ is Burch; see [17, 3.4].
(v) If M is a Burch R-submodule of an R-module T, then $\big(M + xT[x]\big)_{(\mathfrak{m},x)}$ is a Burch S-submodule of the S-module $T[x]_{(\mathfrak{m},x)}$, where $S = R[x]_{(\mathfrak{m},x)}$; see [22, 4.8(2)].
(vi) If R is of prime characteristic p, F-finite, and $\operatorname{depth}(R) = 1$, then ${}^eR$ is Burch for all $e \gg 0$ (here ${}^eR$ denotes the R-module whose underlying abelian group is R, and the R-action on ${}^eR$ is given by $r \cdot x = r^{p^e}x$ for $r \in R$ and $x \in {}^eR$); see 2.2(i) and [44, 3.3].

2.5. A Cohen-Macaulay R-module N is called *Ulrich* [25, 2.1] if the minimal number of generators $\mu_R(N)$ of N equals the (Hilbert-Samuel) multiplicity $\mathrm{e}_R(N)$ of N. Here

$$\mathrm{e}_R(N) = t! \cdot \lim_{n\to\infty} \frac{\operatorname{length}_R(N/\mathfrak{m}^nN)}{n^t}, \text{ where } t = \dim_R(N).$$

Ulrich modules were initially defined in [7] as maximally generated maximal Cohen-Macaulay modules. Such modules were studied extensively in the literature; see, for example, [23] for details.

Let N be a one-dimensional Cohen-Macaulay R-module. It follows from [17, 3.4 and 5.2] that, if N is Ulrich, or equivalently, $N \cong \mathfrak{m}N$, then N is Burch. On the other hand, if N is Burch, then it does not need to be Ulrich. For example, if R is a one-dimensional Cohen-Macaulay ring which does not have minimal multiplicity, that is, $\mathrm{e}(R) > \mu_R(\mathfrak{m})$ [39], and $N = \mathfrak{m}$, then N is Burch but not Ulrich. Similarly, if R is a one-dimensional non-regular Cohen-Macaulay ring and $N = R \oplus \mathfrak{m}$, then N is Burch, but it is not an Ulrich R-module (since R is not Ulrich, or equivalently, since R is not regular).

Finally, we remark that for maximal Cohen-Macaulay Ulrich modules over one-dimensional local domains, the result analogous to Corollary 1.3 follows from [17, 7.4(1) and 5.2].

Recall that R is said to be *Henselian* if each commutative module-finite R-algebra is a direct product of local rings. Hensel's Lemma determines an important class of Henselian rings: If R is complete, then R is Henselian; see [35, 1.9] and also [35, A.30] for a list of equivalent conditions for a local ring to be Henselian.

The following observation plays an important role for the proof of Theorem 2.14.

Lemma 2.6. Let R be a Henselian ring, N an indecomposable rigid R-module, and let $x \in \mathfrak{m}$ be a non zero-divisor on N. Then N/xN is an indecomposable R-module. Therefore, if there is a nonzero cyclic R-module which is a direct summand of N/xN as an R-module, then N is cyclic.

Proof. Note, as $\mathrm{Ext}^1_R(N,N)=0$, the short exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$ gives the short exact sequence $0 \to \mathrm{Hom}_R(N,N) \xrightarrow{x} \mathrm{Hom}_R(N,N) \to \mathrm{Hom}_R(N,N/xN) \to 0$. This yields an isomorphism $\mathrm{Hom}_R(N,N/xN) \cong \mathrm{End}_R(N)/x\mathrm{End}_R(N)$. It follows, since $\mathrm{Hom}_R(N,N/xN) \cong \mathrm{End}_{R/xR}(N/xN)$, that there is a ring isomorphism $\mathrm{End}_{R/xR}(N/xN) \cong \mathrm{End}_R(N)/x\mathrm{End}_R(N)$. Note, since R is Henselian and N is indecomposable, we know that $\mathrm{End}_R(N)$ is a local ring; see [35, 1.8]. So we see that $\mathrm{End}_{R/xR}(N/xN)$ is a local ring. Thus N/xN is an indecomposable R-module.

If there is a nonzero cyclic R-module, say L, which is a direct summand of N/xN as an R-module, then $N/xN \cong L$ so that $\mu_R(N)=\mu_R(N/xN)=\mu_R(L)=1$, that is, N is cyclic, as claimed. $\square$

In general, if the maximal ideal $\mathfrak{m}$ is rigid, that is, if $\mathrm{Ext}^1_R(\mathfrak{m},\mathfrak{m})=0$, then R is regular. This fact follows, for example, from [14, 3.1.1 and 3.1.2]. Here, we provide a brief but alternative justification.

Lemma 2.7. The following implications hold: R is regular if and only if $\mathrm{id}_R(\mathfrak{m})<\infty$ if and only if $\mathrm{Ext}^n_R(\mathfrak{m},\mathfrak{m})=0$ for some $n \geq 0$.

Proof. It suffices to assume $\mathrm{Ext}^n_R(\mathfrak{m},\mathfrak{m})=0$ for some $n \geq 0$, and show that $\mathrm{id}_R(\mathfrak{m})<\infty$; see [45, 4.5, 7.1, and page 659]. We may assume $n \geq 1$. Note, as $\mathrm{Ext}^n_R(\mathfrak{m},\mathfrak{m})=0$, we have that $\mathrm{Ext}^{n+1}_R(k,\mathfrak{m})=0$. If $\mathrm{depth}(R)\geq 1$, then the depth lemma yields $\mathrm{depth}_R(\mathfrak{m})=1$. Moreover, if $\mathrm{depth}(R)=0$, one can check that $\mathrm{depth}_R(\mathfrak{m})=0$. Thus, in either case, $n+1 \geq \mathrm{depth}_R(\mathfrak{m})$. Consequently, we use [38, Thm. 2] along with the vanishing of $\mathrm{Ext}^{n+1}_R(k,\mathfrak{m})=0$, and conclude that $\mathrm{id}_R(\mathfrak{m})<\infty$. $\square$

In passing, we point out that the rigidity hypothesis is needed in Lemma 2.6.

Remark 2.8. Let R be a non-regular domain. Then $\mathfrak{m}$ is not a rigid R-module; see Lemma 2.7. Moreover, since R is a domain, each nonzero proper ideal of R is indecomposable. In particular, $\mathfrak{m}$ is an indecomposable R-module. On the other hand, if $x \in \mathfrak{m}-\mathfrak{m}^2$, then the R-module $\mathfrak{m}/x\mathfrak{m}$ is decomposable; see, for example, [43, 5.3] or [45, 7.11].

Remark 2.9 ([19, 3.8])**.** Let M be an R-module. Then $\bigcup \mathrm{Ass}_R(M) \subseteq \bigcup \mathrm{Ass}(R)$ if and only if M is torsion-free. Moreover, if R has no embedded primes and M is torsion-free, then $M_{\mathfrak{p}}$ is torsion-free over $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Next we prepare a proposition to facilitate the proof of Theorem 2.14. Note that $\mathrm{grade}_R(N)$ denotes the common length of a maximal R-regular sequence in $\mathrm{Ann}_R(N)$ of a given R-module N; see [8, 1.2.6].

Proposition 2.10. *Let R be a Henselian ring and let M be an R-module. Assume M is Burch, rigid, and* $\mathrm{depth}_R(M)\geq 1$*. Then M has a direct summand N such that N is Cohen-Macaulay, cyclic, rigid,* $\mathrm{dim}_R(N)=1$*, and* $\mathfrak{m}N \cong N$*. Moreover, if M is torsion-free, then* $\mathrm{grade}(N)=0$ *and* $\mathrm{depth}(R)\leq 1$.

Proof. Note that, since M is Noetherian, we can write $M \cong \bigoplus_{i=1}^n M_i$ for some nonzero indecomposable R-modules $M_1,\dots,M_n$. As M is Burch and $\mathrm{depth}_R(M)\geq 1$, there is an $x \in \mathfrak{m}$ such that x is a non zero-divisor on M and k is a direct summand of M/xM; see 2.2(i). Therefore we have $M/xM \cong \bigoplus_{i=1}^n M_i/xM_i$. As k is a direct summand of M/xM and k is indecomposable, we see by the Krull–Schmidt theorem [35, 1.8] that k is a direct summand of M_j/xM_j for some j, where $1 \leq j \leq n$. Set $N=M_j$.

As M is rigid and N is a direct summand of M, we see that N is rigid too. Moreover x is a non zero-divisor on N. So N is cyclic by Lemma 2.6. This implies that N/xN is a cyclic R-module and therefore it is an indecomposable R-module. As k is a direct summand of N/xN, we conclude that $k \cong N/xN$. This shows that N is a Cohen-Macaulay R-module with $\dim_R(N) = 1$; see [8, 2.1.2(c) and 2.1.3(a)]. Moreover $\mathfrak{m}(N/xN) = 0$ so that $\mathfrak{m}N = xN \cong N$. This proves the first statement.

Next assume M is torsion-free. Then N is also torsion-free. Therefore, if $\operatorname{grade}_R(N) > 0$, then $N^* = 0$ so that $N = 0$; see [8, 1.2.3(b)]. This proves $\operatorname{grade}_R(N) = 0$.

Let $\mathfrak{p} \in \operatorname{Ass}_R(N)$. Then, as N is torsion-free, there is an $\mathfrak{q} \in \operatorname{Ass}(R)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$; see Remark 2.9. Thus, we have that $\operatorname{depth}(R) \leq \dim(R/\mathfrak{q}) \leq \dim(R/\mathfrak{p}) = \dim_R(N) = 1$; see [8, 1.2.13 and 2.1.2]. □

Proposition 2.10, in view of 2.5, yields the following relation between Burch and Ulrich modules:

Corollary 2.11. *Let R be a Henselian ring and let M be an R-module such that M is Burch, rigid, and $\operatorname{depth}_R(M) \geq 1$. Then M has a direct summand N, where N is a cyclic Ulrich R-module with $\dim_R(N) = 1$.*

We need a few more observations prior to giving a proof of the main theorem of this section.

2.12. Let M be an R-module. Let $\widehat{(-)}$ denote the $\mathfrak{m}$-adic completion functor.

(a) Assume M is Burch, that is, M is an R-submodule of an R-module T such that $\mathfrak{m}(M :_T \mathfrak{m}) \nsubseteq \mathfrak{m}M$. Set $X = \mathfrak{m}(M :_T \mathfrak{m})$ and $Y = \mathfrak{m}M$. If $\widehat{X} \subseteq \widehat{Y}$, then $\widehat{X} \cap \widehat{Y} = \widehat{X \cap Y} = \widehat{X}$ so that $X \cap Y = X$, that is, $X \subseteq Y$. Hence $\widehat{X} \nsubseteq \widehat{Y}$ so that $\widehat{M}$ is Burch over $\widehat{R}$.

(b) If M is torsion-free, or equivalently, if $\bigcup \operatorname{Ass}_R(M) \subseteq \bigcup \operatorname{Ass}(R)$, then we do not know whether or not $\widehat{M}$ must be torsion-free over $\widehat{R}$. On the other hand, we discuss two affirmative cases:

  (i) Assume M is torsion-free and generically free. Then M embeds into a free R-module. Hence, $\widehat{M}$ embeds into a free $\widehat{R}$-module so that $\widehat{M}$ is torsion-free over $\widehat{R}$.
  To see this, consider the exact sequence $0 \to \operatorname{Ext}^1_R(\operatorname{Tr}_R M, R) \to M \xrightarrow{\psi} M^{**}$, where $\operatorname{Tr}_R M$ and M^* denote the Auslander transpose of M and $\operatorname{Hom}_R(M,R)$, respectively; see [40, Prop. 5]. Here, ψ is defined as $\psi(x)(f) = f(x)$ for all $x \in M$ and $f \in M^*$. If $\mathfrak{p} \in \operatorname{Ass}(R)$, then $M_\mathfrak{p}$ is free and hence $\operatorname{Ext}^1_R(\operatorname{Tr}_R M, R)_\mathfrak{p} \cong \operatorname{Ext}^1_{R_\mathfrak{p}}(\operatorname{Tr}_{R_\mathfrak{p}} M_\mathfrak{p}, R_\mathfrak{p}) = 0$; see [40, Pages 5788-5789]. This implies that $\operatorname{Ext}^1_R(\operatorname{Tr}_R M, R)$ is a torsion module. As M is torsion-free and $\operatorname{Ext}^1_R(\operatorname{Tr}_R M, R)$ embeds into M, we see that $\operatorname{Ext}^1_R(\operatorname{Tr}_R M, R) = 0$ and ψ is injective. Pick a free R-module F and a surjection $\pi : F \to M^*$. Then, dualizing π, we obtain $M \overset{\psi}{\hookrightarrow} M^{**} \overset{\pi^*}{\hookrightarrow} F^* \cong F$. This gives an injection $\widehat{M} \hookrightarrow \widehat{F}$, and proves that $\widehat{M}$ is torsion-free over $\widehat{R}$.

  (ii) In general, if $\operatorname{Ass}_R(M) \subseteq \operatorname{Ass}(R)$, then $\operatorname{Ass}_{\widehat{R}}(\widehat{M}) \subseteq \operatorname{Ass}(\widehat{R})$; see [41, 23.2]. Hence, if M is torsion-free and $\operatorname{Ass}(R) = \operatorname{Min}(R)$, then $\operatorname{Ass}_R(M) \subseteq \operatorname{Min}(R)$ so that $\widehat{M}$ is torsion-free over $\widehat{R}$; see Remark 2.9.

(c) Assume M has rank, say r. Let $\mathfrak{q} \in \operatorname{Ass}(\widehat{R})$ and set $\mathfrak{p} = \mathfrak{q} \cap R$. Then $\operatorname{depth}(R_\mathfrak{p}) \leq \operatorname{depth}(\widehat{R}_\mathfrak{q}) = 0$; see [8, 1.2.16]. Hence $\widehat{M}_\mathfrak{q} \cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} \widehat{R}_\mathfrak{q} \cong \widehat{R}_\mathfrak{q}^{\oplus r}$. This shows that $\widehat{M}$ has rank r over $\widehat{R}$.

The next result stems from the arguments of Levin-Vasconcelos [36].

2.13. Let M and N be R-modules such that M has rank and $\operatorname{grade}_R(N) = 0$. If $\operatorname{Ext}^t_R(M, \mathfrak{m}N) = 0$, then $\operatorname{pd}_R(M) < t$; see [17, 2.6(6)].

Next is our main result in this section:

Theorem 2.14. *Let M be an R-module which is Burch, rigid, and torsion-free.*

(i) *If* $\operatorname{Ass}(R) = \operatorname{Min}(R)$, *then* $\operatorname{depth}(R) \leq 1$, *that is, R is either Artinian or* $\operatorname{depth}(R) = 1$.

(ii) *If M has rank, then M is free, and R is either a field or a discrete valuation ring.*

Proof. If $\operatorname{Ass}(R)=\operatorname{Min}(R)$ or M has rank, then $\widehat{M}$ is torsion-free over $\widehat{R}$; see 2.12(b)(ii). Therefore, by considering $\widehat{M}$ over $\widehat{R}$, we may assume R is Henselian and M is a Burch, rigid, and torsion-free R-module; see 2.12(a) and [35, 1.9 and 1.10]. Hence, Proposition 2.10 shows that $\operatorname{depth}(R)\leq 1$ and M has a direct summand N such that $\mathfrak{m}N\cong N$ and $\operatorname{grade}_R(N)=0$.

As M is rigid, we have that $\operatorname{Ext}^1_R(M,N)=\operatorname{Ext}^1_R(M,\mathfrak{m}N)=0$. Hence, if M has rank, then 2.13 implies that M is free. Thus we conclude from 2.2(ii) that R is a field or a discrete valuation ring. □

The rank hypothesis in Theorem 2.14(ii) is necessary; we can see this by the following example.

Example 2.15. Let $R=k[\![x,y]\!]/(xy)$ and $M=R/(x)$. Then $M\cong k[\![y]\!]$. Hence $\operatorname{depth}_R(M)=1$, y is a non zero-divisor on M, and $k\cong M/yM$. Therefore, M is a Burch R-module; see 2.2(ii). Moreover M is rigid, but M does not have rank since $M_{\mathfrak{p}}\neq 0=M_{\mathfrak{q}}$, where $\mathfrak{p}=(x)$ and $\mathfrak{q}=(y)$.

Next we prove Corollary 1.4 advertised in the introduction:

Proof of Corollary 1.4. Note that ${}^{e}R$ is a torsion-free R-module. Note also that, since R is reduced, $R_{\mathfrak{p}}$ is a field for each $\mathfrak{p}\in\operatorname{Ass}(R)$. Hence ${}^{e}R$ has rank for $e\gg 0$; see [34, 2.3 and 2.4]. Hence the claim follows from 2.4(vi) and Theorem 2.14(ii). □

Remark 2.16. As mentioned in the introduction, if R is a one-dimensional F-finite Cohen-Macaulay local ring of prime characteristic p, and if ${}^{e}R$ is rigid for some $e\gg 0$, then it follows from [45, 6.10] that R is regular. The proof of [45, 6.10] relies upon a result of Koh-Lee [33, 2.6]: If R is as before and if $\operatorname{Tor}^R_n(M,{}^{e}R)=0$ for some R-module M and $n\geq 1$, then $\operatorname{pd}_R(M)<\infty$; see also [45, A.3]. Let us note that, under the setup of Corollary 1.4 we do not know whether or not ${}^{e}R$ enjoys a similar property. For example, if $R=\mathbb{F}_p[\![x^4,x^3y,xy^3,y^4]\!]$, then R is an F-finite local domain of depth one and dimension two; in that case [45, 6.10] does not apply, but Corollary 1.4 shows that ${}^{e}R$ is not rigid for all $e\gg 0$.

Recall that, over a one-dimensional Gorenstein domain R, Conjecture 1.1 is equivalent to the following statement: if M is a torsion-free R-module such that $M\otimes_R M^*$ is torsion-free, then M is free. To address the conjecture, Celikbas-Kobayashi [11] proved the following result; see also [17, 7.4(2)] for a similar result for modules.

2.17. ([11, 1.3 and 3.8]) Let R be a one-dimensional local domain which is not regular, and let I and J be ideals of R. If $0\neq I\subseteq\mathfrak{m}J$ and $(I:_R J)=(\mathfrak{m}I:_R\mathfrak{m}J)$, then I is Burch and $I\otimes_R I^*$ has nonzero torsion.

In passing let us note that there are examples of ideals I over one dimensional non-regular local rings R such that $0\neq I\subseteq\mathfrak{m}J$ and $(I:_R J)=(\mathfrak{m}I:_R\mathfrak{m}J)$ for some proper ideal J of R; see, for example, [11, 4.3]. On the other hand, there can be Burch ideals I for which this equality does not hold for any J with $0\neq I\subseteq\mathfrak{m}J$. Next we give such an example. This example indicates that Theorem 1.2 is not subsumed by the result stated in 2.17.

Example 2.18. Let $R=k[\![t^4,t^5,t^6]\!]$ and $I=(t^5,t^8)$. Then $\mathfrak{m}^2=(t^8,t^9,t^{10},t^{11})$ and $\mathfrak{m}I=(t^9,t^{10},t^{11},t^{12})$.

Claim 1: I is Burch, that is, $\mathfrak{m}(I:_R\mathfrak{m})\nsubseteq\mathfrak{m}I$.

Proof of Claim 1: As $\mathfrak{m}^2\subseteq I$, it follows that $\mathfrak{m}\subseteq(I:_R\mathfrak{m})$ and so $\mathfrak{m}=(I:_R\mathfrak{m})$. Thus $\mathfrak{m}(I:_R\mathfrak{m})=\mathfrak{m}^2\nsubseteq\mathfrak{m}I$ as $t^8\in\mathfrak{m}^2$ but $t^8\notin\mathfrak{m}I$.

Claim 2: For each ideal J of R such that $0\neq I\subseteq\mathfrak{m}J$, then $(I:_R J)\neq(\mathfrak{m}I:_R\mathfrak{m}J)$.

Proof of Claim 2: Let J be an ideal of R such that $I\subseteq\mathfrak{m}J$. If $J\neq R$, then $I\subseteq\mathfrak{m}^2$ since $J\subseteq\mathfrak{m}$, but that is not true because $t^5\notin\mathfrak{m}^2$. So $J=R$, and we need to justify $I=(I:_R R)\neq(\mathfrak{m}I:_R\mathfrak{m})$, or equivalently, $(\mathfrak{m}I:_R\mathfrak{m})\nsubseteq I$. We can see this as follows: $t^6\notin I$, but on the other hand $t^6\in(\mathfrak{m}I:_R\mathfrak{m})$ since $t^6\mathfrak{m}=(t^{10},t^{11},t^{12})\subseteq\mathfrak{m}I$.

It is known that Conjecture 1.1 holds for ideals over rings of the form $k[\![t^a,t^{a+1},\dots,t^{2a-2}]\!]$ if k is a field and $a\geq 3$; see [26, 1.6]. Hence, the fact that I in Example 2.18 is not rigid can be deduced by [26, 1.6] or by Theorem 1.2 since it is a Burch ideal.

3. PROOF OF THEOREM 1.5

The main purpose of this section is to prove Theorem 1.5 and formulate Conjecture 1.1 in terms of the vanishing of Ext over Gorenstein local domains. Our argument shows that Conjecture 1.1 is indeed equivalent to a stronger version of a celebrated conjecture of Auslander and Reiten [6]; see the paragraph preceding Theorem 1.5.

The proof of Theorem 1.5 requires some preparation. For that we prove two propositions, each of which deals with modules that have finite Gorenstein dimension.

3.1. ([4]) An R-module M is said to be *totally reflexive* if the natural map $M \to M^{**}$ is bijective and $\mathrm{Ext}^i_R(M,R)=0=\mathrm{Ext}^i_R(M^*,R)$ for all $i \geq 1$. If $M \neq 0$, then the infimum of n for which there exists an exact sequence $0 \to X_n \to \cdots \to X_0 \to M \to 0$, where each X_i is totally reflexive, is called the *Gorenstein dimension* of M. If M has Gorenstein dimension n, we write $\mathrm{G}\text{-}\dim_R(M)=n$. Therefore, M is totally reflexive if and only if $\mathrm{G}\text{-}\dim_R(M) \leq 0$, where it follows by convention that $\mathrm{G}\text{-}\dim_R(0)=-\infty$.

In the rest of the paper we use the following properties freely.

(i) R is Gorenstein if and only if $\mathrm{G}\text{-}\dim_R(M)<\infty$ for each R-module M.
(ii) $\mathrm{G}\text{-}\dim_R(M) \leq \mathrm{pd}_R(M)$, and $\mathrm{G}\text{-}\dim_R(M)=\mathrm{pd}_R(M)$ if $\mathrm{pd}_R(M)<\infty$.
(iii) Assume $M \neq 0$. If $\mathrm{G}\text{-}\dim_R(M)<\infty$, then $\mathrm{G}\text{-}\dim_R(M)+\mathrm{depth}_R(M)=\mathrm{depth}(R)$. Therefore, if R is Cohen-Macaulay and M is totally reflexive, then M is maximal Cohen-Macaulay.
(iv) $\mathrm{G}\text{-}\dim_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \mathrm{G}\text{-}\dim_R(M)$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

The next result is key for our argument.

Proposition 3.2. *Let M be an R-module. Assume $d=\mathrm{depth}(R) \geq 2$ and the following:*
(a) $\mathrm{G}\text{-}\dim_R(M) \leq d-1$.
(b) $\mathrm{pd}_{R_\mathfrak{p}}(M_\mathfrak{p})<\infty$ *for each* $\mathfrak{p} \in \mathrm{Spec}(R)-\{\mathfrak{m}\}$.

Then there exists an exact sequence of R-modules $0 \to Y \to X \to M \to 0$ such that the following hold:
(i) *X is totally reflexive and locally free on* $\mathrm{Spec}(R)-\{\mathfrak{m}\}$.
(ii) $\mathrm{pd}_R(Y) \leq d-2$.
(iii) $\mathrm{Ext}^d_R(M,M) \cong \mathrm{Ext}^d_R(X,X)$.
(iv) *If* $\mathrm{Ext}^{d-1}_R(M,M)=0$*, then* $\mathrm{Ext}^{d-1}_R(X,X)=0$.

Proof. It follows, since $\mathrm{G}\text{-}\dim_R(M)<\infty$, that there is a short exact sequence of R-modules

$$0 \to Y \to X \to M \to 0, \tag{3.2.1}$$

where $\mathrm{pd}_R(Y)<\infty$ and X is totally reflexive; see [5, 1.1] and also [28, 2.10]. As $\mathrm{pd}_{R_\mathfrak{p}}(M_\mathfrak{p})<\infty$ for each $\mathfrak{p} \in \mathrm{Spec}(R)-\{\mathfrak{m}\}$, it follows that X is locally free on $\mathrm{Spec}(R)-\{\mathfrak{m}\}$. Note also $\mathrm{depth}_R(M) \geq 1$ since $\mathrm{G}\text{-}\dim_R(M) \leq d-1$.

If $\mathrm{depth}_R(M) \geq d$, then the depth lemma applied to the exact sequence $0 \to Y \to X \to M \to 0$ shows that $\mathrm{depth}_R(Y) \geq d$; so, in this case, Y is free since $\mathrm{pd}_R(Y)<\infty$. On the other hand, if $\mathrm{depth}_R(M) \leq d-1$, then, since $\mathrm{depth}_R(M) \geq 1$, we can use the depth lemma once more along with the exact sequence (3.2.1) and see that $\mathrm{depth}_R(Y)=\mathrm{depth}_R(M)+1 \geq 2$. Hence, regardless of what the depth of M is, we have that $\mathrm{pd}_R(Y) \leq d-2$.

The exact sequence (3.2.1) yields the following exact sequences for each $i \geq 1$:

$$\mathrm{Ext}^i_R(X,Y) \to \mathrm{Ext}^i_R(X,X) \to \mathrm{Ext}^i_R(X,M) \to \mathrm{Ext}^{i+1}_R(X,Y), \tag{3.2.2}$$

and

$$\mathrm{Ext}^i_R(M,M) \to \mathrm{Ext}^i_R(X,M) \to \mathrm{Ext}^i_R(Y,M) \to \mathrm{Ext}^{i+1}_R(M,M) \to \mathrm{Ext}^{i+1}_R(X,M) \to \mathrm{Ext}^{i+1}_R(Y,M). \tag{3.2.3}$$

Note that $\mathrm{Ext}^j_R(X,Y)=0$ for all $j \geq 1$ since $\mathrm{pd}_R(Y)<\infty$ and X is totally reflexive; see [4, 4.12]. So (3.2.2) gives:

$$\mathrm{Ext}^i_R(X,X) \cong \mathrm{Ext}^i_R(X,M) \text{ for each } i \geq 1. \tag{3.2.4}$$

Note that, since $\mathrm{pd}_R(Y) \le d-2$, we have $\mathrm{Ext}_R^{d-1}(Y,M) = \mathrm{Ext}_R^d(Y,M) = 0$. Hence, setting $i = d-1$ in (3.2.3), we obtain from (3.2.3) and (3.2.4) that:

$$\mathrm{Ext}_R^d(M,M) \cong \mathrm{Ext}_R^d(X,M) \cong \mathrm{Ext}_R^d(X,X). \tag{3.2.5}$$

Now assume $\mathrm{Ext}_R^{d-1}(M,M) = 0$. Then, as $\mathrm{Ext}_R^{d-1}(Y,M) = 0$, we see from (3.2.3) that $\mathrm{Ext}_R^{d-1}(X,M)$ vanishes. So (3.2.4) shows that $\mathrm{Ext}_R^{d-1}(X,X) = 0$, as required. □

We recall a few facts for the proof of Proposition 3.6.

3.3. An R-module M is said to satisfy *Serre's condition* (S_n) for some integer $n \ge 0$ provided that $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \ge \min\{n, \dim(R_\mathfrak{p})\}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$; see [20, page 3] (note that there are different versions of Serre's condition; cf. [8, page 63]).

If R is Gorenstein, then M satisfies (S_2) if and only if M is reflexive if and only if M is a second syzygy module; see [20, 3.6]

3.4. Let M be a nonzero R-module. If $\mathrm{pd}_R(M) = n < \infty$, then $\mathrm{Ext}_R^n(M,M) \neq 0$; see [41, Lemma 1(iii), page 154].

3.5. Let R be a d-dimensional Cohen-Macaulay local ring with canonical module ω_R, and let M and N be R-modules. If N is maximal Cohen-Macaulay such that M or $N^\dagger$ is locally free on $\mathrm{Spec}(R) - \{\mathfrak{m}\}$, then $\mathrm{Ext}_R^i(M,N) \cong \mathrm{Ext}_R^i(N^\dagger \otimes_R M, \omega_R)$ for all $i = 0, \dots, d$, where $N^\dagger = \mathrm{Hom}_R(N, \omega_R)$; see [24, 2.3].

Proposition 3.6. *Let R be a d-dimensional Cohen-Macaulay local ring with a canonical module ω_R. Consider the following two conditions on R:*

(i) *Given a totally reflexive R-module M, if M has rank and $M \otimes_R M^\dagger$ satisfies (S_2), then M is free.*
(ii) *Given a torsion-free R-module N, if N has rank,* $\mathrm{G\text{-}dim}_R(N) < \infty$, *and* $\mathrm{Ext}_R^i(N,N) = 0$ *for all $i = 1, \dots, d$, then N is free.*

If $R_\mathfrak{p}$ satisfies the condition in part (i) for all $\mathfrak{p} \in \mathrm{Spec}(R)$, then R satisfies the condition in part (ii).

Proof. Assume $R_\mathfrak{p}$ satisfies the condition in part (i) for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Let N be a torsion-free R-module with rank such that $\mathrm{G\text{-}dim}_R(N) < \infty$ and $\mathrm{Ext}_R^i(N,N) = 0$ for all $i = 1, \dots, d$. We proceed by induction on d to prove that N is free.

If $d = 0$, then N is free since it has rank. So we assume $d = 1$. Then, since N is totally reflexive and locally free on $\mathrm{Spec}(R) - \{\mathfrak{m}\}$, it follows from 3.5 that $\mathrm{Ext}_R^1(N,N) \cong \mathrm{Ext}_R^1(N \otimes_R N^\dagger, \omega_R)$. Therefore $\mathrm{Ext}_R^1(N \otimes_R N^\dagger, \omega_R) = 0$. Hence $N \otimes_R N^\dagger$ is maximal Cohen-Macaulay and so it satisfies (S_2); see [8, 3.5.11(b)] and 3.3. Consequently, N is free by the hypothesis.

Next assume $d \ge 2$. Then, by using the induction hypothesis, we may assume that N is locally free on $\mathrm{Spec}(R) - \{\mathfrak{m}\}$. It follows from Proposition 3.2 that there is an exact sequence $0 \to Y \to X \to N \to 0$, where X is totally reflexive and $\mathrm{Ext}_R^d(X,X) = \mathrm{Ext}_R^{d-1}(X,X) = 0$. Note that, since X is totally reflexive, it is maximal Cohen-Macaulay; see 3.1(iii). Thus, by 3.5, we have that

$$\mathrm{Ext}_R^i(X \otimes_R X^\dagger, \omega_R) \cong \mathrm{Ext}_R^i(X,X) \text{ for } i = d-1 \text{ and } i = d.$$

Hence [8, 3.5.11(b)] implies that $\mathrm{depth}_R(X \otimes_R X^\dagger) \ge 2$. As X is locally free on $\mathrm{Spec}(R) - \{\mathfrak{m}\}$, we see that $X \otimes_R X^\dagger$ satisfies (S_2). So X is free by the hypothesis (i). Consequently, as $\mathrm{pd}_R(Y) < \infty$, it follows that $\mathrm{pd}_R(N) < \infty$. Also, as $\mathrm{Ext}_R^i(N,N) = 0$ for all $i = 1, \dots, d$, we conclude by 3.4 that N is free. □

If R is a Noetherian integrally closed domain (not necessarily local) and M is a torsion-free R-module such that $M \otimes_R M^*$ is reflexive, then Auslander [3, 3.3] proved that M must be projective. In the local case, his argument actually establishes the following result, for which we provide a proof for the convenience of the reader; see also [12, 8.6].

3.7. (Auslander [3]) Let R be a local ring satisfying (S_2) and let M be a torsion-free R-module such that $M \otimes_R M^*$ is reflexive. If $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for each prime ideal $\mathfrak{p}$ of R of height at most one, then M is free.

Proof. Set $X = M \otimes_R M^*$ and $Y = \operatorname{Hom}_R(M,M)$. We consider the map $X \xrightarrow{\alpha} Y$, where α is defined as $\big(\alpha(x \otimes f)\big)(z) = f(z)x$ for all $x, z \in M$ and $f \in M^*$. Setting $A = \ker(\alpha)$ and $B = \operatorname{coker}(\alpha)$, we have an exact sequence $0 \to A \to X \xrightarrow{\alpha} Y \to B \to 0$. Here, B is the quotient of Y by the submodule of all homomorphisms $M \to M$ that factor through a free R-module. Hence, α is surjective if and only if M is free; see, for example, [42, 3.8].

Let $\mathfrak{p} \in \operatorname{Ass}(R)$. Then, since $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$, it follows that $\alpha_{\mathfrak{p}}$ is surjective and so $B_{\mathfrak{p}} = 0$. Hence, we have an exact sequence $0 \to A_{\mathfrak{p}} \to X_{\mathfrak{p}} \xrightarrow{\alpha_{\mathfrak{p}}} Y_{\mathfrak{p}} \to 0$. As $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ are free modules over $R_{\mathfrak{p}}$ of the same rank, we deduce that $A_{\mathfrak{p}} = 0$. Hence, A is torsion. Thus, $A = 0$ since X is a torsion-free R-module.

Suppose $B \neq 0$ and pick $\mathfrak{q} \in \operatorname{Ass}_R(B)$. As $B_{\mathfrak{q}} \neq 0$ and $\alpha_{\mathfrak{p}}$ is surjective for each prime ideal $\mathfrak{p}$ of R of height at most one, it follows that $\dim(R_{\mathfrak{q}}) \geq 2$. Therefore, $\operatorname{depth}(R_{\mathfrak{q}}) \geq 2$ as R satisfies (S_2). Hence, $X_{\mathfrak{q}}$, being reflexive over $R_{\mathfrak{q}}$, has depth at least two. Note also that, since M is torsion-free, so is Y. Furthermore, $Y_{\mathfrak{p}}$ is torsion-free over $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ because R has no embedded primes; see [19, 3.8]. Now, since $\operatorname{depth}_{R_{\mathfrak{q}}}(Y_{\mathfrak{q}}) \geq 1 > 0 = \operatorname{depth}_{R_{\mathfrak{q}}}(B_{\mathfrak{q}})$, the depth lemma applied to the short exact sequence $0 \to X_{\mathfrak{q}} \to Y_{\mathfrak{q}} \to B_{\mathfrak{q}} \to 0$ implies that $\operatorname{depth}_{R_{\mathfrak{q}}}(X_{\mathfrak{q}}) = \operatorname{depth}_{R_{\mathfrak{q}}}(B_{\mathfrak{q}}) + 1 = 1$. This contradiction shows that $B = 0$. □

We are now ready to give a proof of Theorem 1.5:

Proof of Theorem 1.5. Assume first condition (b) holds. Let R be a one-dimensional Gorenstein local domain and let N be a torsion-free R-module such that $N \otimes_R N^*$ is torsion-free. Then it follows from 3.5 that $\operatorname{Ext}^1_R(N,N) = 0$. Therefore, by our assumption, N is free. This shows that Conjecture 1.1 holds over each Gorenstein domain.

Next assume condition (a) holds, namely, Conjecture 1.1 holds over each Gorenstein local domain. Let R be a d-dimensional Gorenstein local domain, and let M be a torsion-free R-module such that $\operatorname{Ext}^i_R(M,M) = 0$ for all $i = 1, \ldots, d$. We claim that M is free.

Note that $\text{G-dim}_R(M) < \infty$ as R is Gorenstein. Therefore M has all the properties stated in part (ii) of Proposition 3.6. Consequently, to show M is free, it suffices to prove that $R_{\mathfrak{p}}$ satisfies the condition of part (i) of Proposition 3.6 for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

For that let $\mathfrak{p} \in \operatorname{Spec}(R)$ and set $S = R_{\mathfrak{p}}$. Note that each module over S has rank since S is a domain. Let X be a totally reflexive S-module such that $X \otimes_S X^*$ satisfies (S_2). Then, since S is Gorenstein, X is a maximal Cohen-Macaulay S-module and $X \otimes_S X^*$ is reflexive. It suffices to show that X is a free S-module.

If $\dim(S) \leq 0$, then S is a field and hence X is free. If $\dim(S) = 1$, then X is free because it is assumed that Conjecture 1.1 holds over each one-dimensional local Gorenstein domain and S is such a ring. Next suppose $\dim(S) \geq 2$. Then, by the induction hypothesis on $\dim(S)$, we conclude that $X_{\mathfrak{q}}$ is a free $S_{\mathfrak{q}}$-module for each prime ideal $\mathfrak{q}$ of S of height at most one. So 3.7 shows that X is free. □

Appendix A. On a result of Araya

In this appendix we state some byproducts of our argument that are not directly related to Conjecture 1.1. In particular, we prove Proposition 1.7 which generalizes the following result of Araya:

A.1. ([1, Corollary 10]) Let R be a d-dimensional Gorenstein local ring, where $d \geq 1$, and let M be an R-module. Assume the following hold:

(i) M is locally free on $\operatorname{Spec}(R) - \{\mathfrak{m}\}$.
(ii) M is maximal Cohen-Macaulay.
(iii) $\operatorname{Ext}^{d-1}_R(M,M) = 0$.

Then M is free.

Araya's result [1] shows that the Auslander-Reiten conjecture is straightforward over Gorenstein isolated singularities. It also implies that the conjecture holds over Gorenstein normal domains, a result initially proved by Huneke and Jorgensen [30, 5.9]. Subsequently Kimura [32] removed the Gorenstein hypothesis, and proved that the Auslander-Reiten conjecture holds over arbitrary normal domains. In his paper Kimura also proved:

A.2. ([32, 2.8 and 2.10(3)]) Let R be a local ring such that $d=\operatorname{depth}(R)\geq 1$, and let X be an R-module such that $X_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p}\in\operatorname{Spec}(R)-\{\mathfrak{m}\}$ and $\operatorname{Ext}^{d-1}_R(X,X)=0$. Then X is free provided that at least one of the following holds:

(i) $\operatorname{Ext}^i_R(X,R)=0$ for all $i=1,\dots,2d+1$.
(ii) R satisfies (S_2), $\operatorname{Ext}^i_R(X,R)=0$ for all $i=1,\dots u$, and $\operatorname{Ext}^i_R(\operatorname{Tr}_R X,R)=0$ for all $i=1,\dots v$ for some integers $u\geq 0$ and $v\geq 0$ such that $u+v=2d+1$.

Next we use Proposition 3.2 and A.2(i), and give a proof for Proposition 1.7 which is advertised in the introduction:

Proof of Proposition 1.7. We may assume $d\geq 2$ and $M\neq 0$. As M is torsion-free, it follows that $\operatorname{depth}_R(M)\geq 1$ and hence $\operatorname{G-dim}_R(M)\leq d-1$. Then Proposition 3.2 gives an exact sequence $0\to Y\to X\to M\to 0$, where X is totally reflexive, X is locally free on $\operatorname{Spec}(R)-\{\mathfrak{m}\}$, and $\operatorname{Ext}^{d-1}_R(X,X)=0$. Thus A.2 implies that X is free. Hence $\operatorname{pd}_R(M)\leq d-1$. Moreover, since $\operatorname{Ext}^{d-1}_R(M,M)=0$, we see from 3.4 that $\operatorname{pd}_R(M)\neq d-1$. □

A consequence of Proposition 1.7 is:

Corollary A.3. *Let R be a d-dimensional local Gorenstein ring, with $d\geq 1$, and let I be an $\mathfrak{m}$-primary ideal of R. Then $\operatorname{Ext}^{d-1}_R(I,I)\neq 0$.*

Proof. As $d\geq 1$ and I is an $\mathfrak{m}$-primary ideal of R, we have that $I\neq 0$. So the natural short exact sequence $0\to I\to R\to R/I\to 0$ implies that $\operatorname{depth}_R(I)=1$ since $d\geq 1$ and R/I is a finite length R-module.

Suppose $\operatorname{Ext}^{d-1}_R(I,I)=0$. As I is a torsion-free R-module and $I_{\mathfrak{p}}\cong R_{\mathfrak{p}}$ for all $\mathfrak{p}\in\operatorname{Spec}(R)-\{\mathfrak{m}\}$, we use Proposition 1.7 and conclude that $\operatorname{pd}_R(I)\leq d-2$. In that case, the Auslander-Buchsbaum formula implies that $\operatorname{depth}_R(I)\geq 2$, which is not possible. Therefore, $\operatorname{Ext}^{d-1}_R(I,I)\neq 0$. □

In general we do not know whether or not M must be free if R is a d-dimensional Gorenstein local domain and M is a torsion-free R-module such that $\operatorname{Ext}^i_R(M,M)=0$ for all $i=1,\dots,d$; this is the context of Theorem 1.5. However, if M is locally free on the set $\mathsf{X}^1(R)$ of all prime ideals $\mathfrak{p}$ of R with $\dim(R_{\mathfrak{p}})\leq 1$, then the vanishing of $\operatorname{Ext}^i_R(M,M)$ for all $i=1,\dots,d-1$ is sufficient to conclude that M is free. We record this observation which also extends [2, 1.6] for the case where $n=1$.

Corollary A.4. *Let R be a d-dimensional local ring satisfying (S_1), where $d\geq 1$, and let M be an R-module. Assume the following hold:*

(i) *$M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for all $\mathfrak{p}\in\mathsf{X}^1(R)$.*
(ii) *M is torsion-free.*
(iii) *$\operatorname{Ext}^i_R(M,M)=0$ for $i=1,\dots,d-1$.*

If $\operatorname{G-dim}_R(M)<\infty$, then M is free.

Proof. There is nothing to prove if $d=1$. Assume $d\geq 2$. Then $M_{\mathfrak{p}}$ is torsion-free over $R_{\mathfrak{p}}$ since R satisfies (S_1) [19, 3.8]. Hence, by the induction hypothesis on d, we have that $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for all $\mathfrak{p}\in\operatorname{Spec}(R)-\{\mathfrak{m}\}$. So Proposition 1.7 shows that $\operatorname{pd}_R(M)\leq\operatorname{depth}(R)-2$. As $\operatorname{Ext}^i_R(M,M)=0$ for $i=1,\dots,d-1$, we conclude that M is free; see 3.4. □

We prove an analog of Proposition 1.7 by replacing the finite Gorenstein dimension hypothesis on M with the assumption of the vanishing of $\operatorname{Ext}^i_R(M,R)$ for $i=1,\dots,d$ if R satisfies (S_2) and M is locally free on $\operatorname{Spec}(R)-\{\mathfrak{m}\}$; see Corollary A.7. This follows from a more general result which we prove as Proposition A.6; see also [27, 3.14]. In the proof we use A.2(ii) and the following auxiliary result:

A.5. Let R be a local ring, M and N be R-modules, and let $n \geq 1$. Assume $\operatorname{Ext}_R^{n+1}(M,R) = 0$. Then the canonical map $\operatorname{Ext}_R^n(M,N) \to \operatorname{Ext}_R^n(\Omega M, \Omega N)$ induced by the syzygy sequence is surjective. Moreover, if $\operatorname{Ext}_R^n(M,R) = 0$, then $\operatorname{Ext}_R^n(M,N) \cong \operatorname{Ext}_R^n(\Omega M, \Omega N)$; see, for example, [32, 2.6].

Proposition A.6. *Let R be a local ring, with $d = \operatorname{depth}(R)$, M be an R-module, and let $n \geq 0$ be an integer such that $d \geq n+1$. Assume the following hold:*

(i) *R satisfies (S_2).*
(ii) *$M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ and $\operatorname{depth}_R(M) \geq n$.*
(iii) $\operatorname{Ext}_R^{d-1}(M,M) = 0$.
(iv) $\operatorname{Ext}_R^i(M,R) = 0$ *for* $\min\{d, d-n+2\} \leq i \leq 2d-n+1$.

Then $\operatorname{pd}_R(M) < \infty$.

Proof. The case where $d = 1$ is trivial since we assume $\operatorname{Ext}_R^{d-1}(M,M) = 0$. Hence we assume $d \geq 2$.

Note, since $d \geq \min\{d, d-n+2\}$, part (iv) implies:

$$\operatorname{Ext}_R^i(M,R) = 0 \text{ for } i = d, \dots, 2d-n. \tag{A.6.1}$$

Therefore, in view of (A.6.1), the observation recorded in A.5 yields:

$$\operatorname{Ext}_R^{d-1}(M,M) \twoheadrightarrow \operatorname{Ext}_R^{d-1}(\Omega_R M, \Omega_R M) \cong \operatorname{Ext}_R^{d-1}(\Omega_R^2 M, \Omega_R^2 M) \cong \cdots \cong \operatorname{Ext}_R^{d-1}(\Omega_R^{d-n+1} M, \Omega_R^{d-n+1} M).$$

As $\operatorname{Ext}_R^{d-1}(M,M) = 0$, we conclude that $\operatorname{Ext}_R^{d-1}(\Omega_R^{d-n+1} M, \Omega_R^{d-n+1} M) = 0$. We set $X = \Omega_R^{d-n+1} M$. Then it follows that

$$\operatorname{Ext}_R^{d-1}(X,X) = 0. \tag{A.6.2}$$

We deduce from part (iv) that:

$$\operatorname{Ext}_R^{i+d-n+1}(M,R) \cong \operatorname{Ext}_R^i(X,R) = 0 \text{ for all } i = 1, \dots, d. \tag{A.6.3}$$

As $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ and $\operatorname{depth}_R(M) \geq n$, it follows that M is an nth syzygy module; see [18, 2.4]. So X is a $(d+1)$st syzygy module. Moreover $X_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ since so is $M_{\mathfrak{p}}$. Thus [40, Thm. 43] implies that

$$\operatorname{Ext}_R^i(\operatorname{Tr}_R X, R) = 0 \text{ for all } i = 1, \dots, d+1. \tag{A.6.4}$$

Finally we make use of (A.6.2), (A.6.3), and (A.6.4), and conclude from Kimura's result A.2(ii) that X is free. Consequently $\operatorname{pd}_R(M) < \infty$. □

Next is a corollary of Proposition A.6 which establishes a variation of Proposition 1.7.

Corollary A.7. *Let R be a local ring satisfying (S_2), with $d = \operatorname{depth}(R) \geq 1$, and let M be an R-module. Assume the following hold:*

(i) *$M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$.*
(ii) *M is torsion-free.*
(iii) $\operatorname{Ext}_R^{d-1}(M,M) = 0$.

If $\operatorname{Ext}_R^i(M,R) = 0$ *for all $i = d, \dots, 2d+1$, then* $\operatorname{pd}_R(M) \leq d-2$.

Proof. We may assume $M \neq 0$, and we can use Proposition A.6 for the case where $n = 0$ and conclude that $\operatorname{pd}_R(M) < \infty$. It follows, as $d = \operatorname{depth}(R) \geq 1$ and M is torsion-free, that $\operatorname{depth}_R(M) \geq 1$ (recall that each non zero-divisor on R is a non zero-divisor on M). Therefore, the Auslander-Buchsbaum formula shows that $\operatorname{pd}_R(M) \leq d-1$. Now, since $\operatorname{Ext}_R^{d-1}(M,M) = 0$, 3.4 shows that $\operatorname{pd}_R(M) \leq d-2$. □

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REFERENCES

1. Tokuji Araya, *The Auslander-Reiten conjecture for Gorenstein rings*, Proc. Amer. Math. Soc. **137** (2009), no. 6, 1941–1944. 3, 9
2. Tokuji Araya, Olgur Celikbas, Arash Sadeghi, and Ryo Takahashi, *On the vanishing of self extensions over Cohen-Macaulay local rings*, Proc. Amer. Math. Soc. **146** (2018), no. 11, 4563–4570. 10
3. Maurice Auslander, *Modules over unramified regular local rings*, Illinois J. Math. **5** (1961), 631–647. 8
4. Maurice Auslander and Mark Bridger, *Stable module theory*, Mem. Amer. Math. Soc., No. 94, American Mathematical Society, Providence, R.I., 1969. 7
5. Maurice Auslander and Ragnar-O. Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Mém. Soc. Math. France (N.S.) (1989), no. 38, 5–37. 7
6. Maurice Auslander and Idun Reiten, *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. **52** (1975), 69–74. 2, 7
7. Joseph P. Brennan, Jürgen Herzog, and Bernd Ulrich, *Maximally generated Cohen-Macaulay modules*, Math. Scand. **61** (1987), no. 2, 181–203. 3
8. Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. 4, 5, 8
9. Olgur Celikbas, *Vanishing of Tor over complete intersections*, J. Commut. Algebra **3** (2011), no. 2, 169–206. 1
10. Olgur Celikbas, Shiro Goto, Ryo Takahashi, and Naoki Taniguchi, *On the ideal case of a conjecture of Huneke and Wiegand*, Proc. Edinb. Math. Soc. (2) **62** (2019), no. 3, 847–859. 1
11. Olgur Celikbas and Toshinori Kobayashi, *On a class of Burch ideals and a conjecture of Huneke and Wiegand*, Collect. Math. **73** (2022), no. 2, 221–236. 3, 6
12. Olgur Celikbas and Roger Wiegand, *Vanishing of Tor, and why we care about it*, J. Pure Appl. Algebra **219** (2015), no. 3, 429–448. 1, 2, 8
13. Hailong Dao and David Eisenbud, *Burch index, summands of syzygies and linearity in resolutions*, Bull. Iran. Math. Soc. 49, 10 (2023). 1
14. Hailong Dao, *Some homological properties of modules over a complete intersection, with applications*, Peeva, Irena (ed.), Commutative algebra. Expository papers dedicated to David Eisenbud on the occasion of his 65th birthday. New York, NY: Springer, 335–371 (2013). 4
15. Hailong Dao, Toshinori Kobayashi, and Ryo Takahashi, *Burch ideals and Burch rings*, Algebra Number Theory **14** (2020), no. 8, 2121–2150. 1
16. Michael DeBellevue and Claudia Miller, *Summands of syzygies over rings of positive Burch index via canonical resolutions*, preprint; posted at arXiv:2401.00142 (2023). 1
17. Souvik Dey and Toshinori Kobayashi, *Vanishing of (co)homology of Burch and related submodules*, Illinois J. Math. **67** (2023), no. 1, 101–151. 1, 2, 3, 5, 6
18. Mohammad T. Dibaei and Arash Sadeghi, *Linkage of modules and the Serre conditions*, J. Pure Appl. Algebra **219** (2015), no. 10, 4458–4478. 11
19. Neil Epstein and Yongwei Yao, *Criteria for flatness and injectivity*, Math. Z. **271** (2012), no. 3-4, 1193–1210. 1, 4, 9, 10
20. E. Graham Evans and Phillip Griffith, *Syzygies*, London Mathematical Society Lecture Note Series, vol. 106, Cambridge University Press, Cambridge, 1985. 8
21. Pedro A. García-Sánchez and Micah J. Leamer, *Huneke-Wiegand Conjecture for complete intersection numerical semigroup rings*, J. Algebra **391** (2013), 114–124. 1
22. Dipankar Ghosh and Aniruddha Saha, *Homological dimensions of Burch ideals, submodules and quotients*, J. Pure Appl. Algebra 228 (2024), no. 7, Paper No. 107647. 1, 3
23. Shiro Goto, Kazuho Ozeki, Ryo Takahashi, Kei-Ichi Watanabe, and Ken-Ichi Yoshida, *Ulrich ideals and modules*, Math. Proc. Cambridge Philos. Soc., vol. 156, Cambridge University Press, 2014, pp. 137–166. 3
24. Shiro Goto and Ryo Takahashi, *On the Auslander-Reiten conjecture for Cohen-Macaulay local rings*, Proc. Amer. Math. Soc. **145** (2017), no. 8, 3289–3296. 8

25. Shiro Goto, Ryo Takahashi, and Naoki Taniguchi, *Almost Gorenstein rings—towards a theory of higher dimension*, J. Pure Appl. Algebra **219** (2015), no. 7, 2666–2712. 3
26. Shiro Goto, Ryo Takahashi, Naoki Taniguchi, and Hoang Le Truong, *Huneke-Wiegand conjecture and change of rings*, J. Algebra **422** (2015), 33–52. 1, 6
27. Rafael Holanda and Cleto B. Miranda-Neto, *Vanishing of (co)homology, freeness criteria, and the Auslander-Reiten conjecture for Cohen-Macaulay Burch rings*, Preprint; posted at arXiv.2212.05521, 2022. 10
28. Henrik Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193. 7
29. Craig Huneke, Srikanth B. Iyengar, and Roger Wiegand, *Rigid ideals in Gorenstein rings of dimension one*, Acta Math. Vietnam. **44** (2019), no. 1, 31–49. 1
30. Craig Huneke and David A. Jorgensen, *Symmetry in the vanishing of Ext over Gorenstein rings*, Math. Scand. **93** (2003), no. 2, 161–184. 10
31. Craig Huneke and Roger Wiegand, *Tensor products of modules and the rigidity of Tor*, Math. Ann. **299** (1994), no. 3, 449–476. 1
32. Kaito Kimura, *Auslander-Reiten conjecture for normal rings*, Preprint, posted at arXiv:2304.03956 (2023). 2, 10, 11
33. Jee Koh and Kisuk Lee, *Some restrictions on the maps in minimal resolutions*, J. Algebra **202** (1998), no. 2, 671–689. 6
34. Ernst Kunz, *On Noetherian rings of characteristic p*, Amer. J. Math. **98** (1976), no. 4, 999–1013. 6
35. Graham J. Leuschke and Roger Wiegand, *Cohen-Macaulay representations*, Math. Surveys Monogr., vol. 181, American Mathematical Society, Providence, RI, 2012. 4, 6
36. Gerson Levin and Wolmer V. Vasconcelos, *Homological dimensions and Macaulay rings*, Pacific J. Math. **25** (1968), 315–323. 5
37. Haydee Lindo, *Trace ideals and centers of endomorphism rings of modules over commutative rings*, J. Algebra **482** (2017), 102–130. 1
38. Paul Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. **(4)**, 9(1):103–106, 1976. 4
39. Shreeram Shankar Abhyankar, *Local rings of high embedding dimension*, Amer. J. Math. **89** (1967), 1073–1077. 3
40. Vladimir Maşiek, *Gorenstein dimension and torsion of modules over commutative Noetherian rings*, Commun. Algebra **28** (2000), no. 12, 5783–5811. 5, 11
41. Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. 5, 8
42. Yoshino, Yuji. *Cohen-Macaulay modules over Cohen-Macaulay rings*. London Mathematical Society Lecture Note Series, 146. Cambridge University Press, Cambridge, 1990. 9
43. Ryo Takahashi, *Syzygy modules with semidualizing or G-projective summands*, J. Algebra **295** (2006), no. 1, 179–194. 4
44. Ryo Takahashi and Yuji Yoshino, *Characterizing Cohen-Macaulay local rings by Frobenius maps*, Proc. Amer. Math. Soc. **132** (2004), no. 11, 3177–3187. 3
45. Majid Rahro Zargar, Olgur Celikbas, Mohsen Gheibi, and Arash Sadeghi, *Homological dimensions of rigid modules*, Kyoto J. Math. **58** (2018), no. 3, 639–669. 2, 4, 6